SUB- AND SUPER-ALFVÉNIC FLOWS PAST BODIES*

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Summary—In steady magnetohydrodynamic flows past obstacles an important parameter is the "Alfvén number" m, defined as the ratio of stream speed to the speed of Alfvén waves. To illustrate the significance of this parameter, the particular category of flow is considered in which the undisturbed velocity and magnetic-field vectors far from the obstacle are parallel or anti-parallel, i.e. "aligned-fields" flow. For such flows the change from $m>1$ (super-Alfvénic) to $m<1$ (sub-Alfvénic) is especially profound.

Contrasts between super- and sub-Alfvénic flows are described in terms of the following categories of sub-Alfvénic phenomena: (1) upstream wakes, (2) upstream-inclined waves, and (3) elliptic supersonic and hyperbolic subsonic flows; negative lift of airfoils at positive incidence. These phenomena are discussed with a view toward explaining their origins. It is concluded that none violates simple physical principles, but that the lift produced in elliptic sub-Alfvénic regimes cannot be predicted with confidence until the analog of the Kutta-Joukowski condition is understood.

Finally, an attempt is made to assess the probability of laboratory observations of these phenomena. It is concluded that values of the magnetic Reynolds number can be realized that will permit these effects to be studied before they are attenuated by diffusion.

INTRODUCTION

In studies of steady flow of electrically conducting fluids, a dimensionless parameter of great significance in determining the character of the flow is the ratio of the magnetic pressure to the dynamic pressure,

$$m^{-2} = \frac{H^2/8\pi}{\varrho q^2/2} \tag{1}$$

This ratio of pressures is also the square of a ratio of speeds, since

$$A = H/\sqrt{4\pi\varrho} \tag{2}$$

is the propagation speed of small magnetohydrodynamic waves, the so-called Alfvén waves. Thus m is a ratio quite analogous to the Mach number; i.e. the ratio of fluid speed to the speed of certain small disturbances.

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We propose to call this ratio, $m = U/A$, the Alfvén number, although some other writers seem to prefer the name Alfvén Mach number.

In view of this analogy, it is not surprising to an aerodynamicist that the Alfvén number plays an important role in determining the character of magnetohydrodynamic flow and that profound differences in character are predicted between flows where $m < 1$ and those where $m > 1$. These predicted differences are most striking, perhaps, in flows of conducting fluids past solid obstacles, i. e. stream flows, in which the magnetic field as well as the velocity field, is supposed to be uniform at large distances from the body. In such cases the disturbances created by an obstacle tend to propagate relative to the fluid as Alfvén waves along the magnetic lines, and changes of the Alfvén number can change significantly the direction of this resultant propagation.

In particular, several investigators[1–14, 24] have studied the special case in which the magnetic and velocity fields are not only both uniform at infinity but are also parallel (or anti-parallel) there; this is usually called "aligned fields". Changes of flow character of particular interest in these flows occur for a velocity change from super-Alfvénic ($m > 1$) to sub-Alfvénic ($m < 1$). These contrasts are the subject of the present paper.

In the literature of this field we can find at least three striking contrasts between sub- and super-Alfvénic aligned flows. In naming and describing them we naturally emphasize the phenomena of sub-Alfvénic flow, since the category of super-Alfvénic flows includes the regime of conventional fluid mechanics at the limit $m \to \infty$.

1. *Upstream Wakes and Reversed Boundary Layers*

Greenspan and Carrier[1], Hasimoto[2, 11], Lary[4], Imai[12], Gourdine[13, 14], Yosinobu[24], and others have noted that steady incompressible magnetohydrodynamic flows rather generally involve wake-like phenomena extending outward from obstacles, and that these may, at sub-Alfvénic speeds, extend upstream*. In fluids of non-zero viscosity and electrical resistance there appear to be two wakes, one extending upstream and the other downstream. Surprisingly, however, in the flow of a perfect conductor (infinite conductivity), the downstream wake disappears and the upstream remains, in sub-Alfvénic flow. By means of an elegant proof,

* It seems to us that the term "wake" is used too freely by some of these authors[11, 12, 15]. Except in the special case of aligned fields, the diffusion phenomena that they consider do not represent diffusion along streamlines and therefore should be called "diffuse waves" rather than "wakes". Moreover, it seems to have been overlooked that a true wake, i.e. diffusion along streamlines, still occurs, in addition to the phenomena discussed. In the aligned-field case discussed here, the true wake coincides with one family of diffuse waves.

Hasimoto has shown[2] that the steady, sub-Alfvénic, aligned-fields flow of a viscous perfect conductor about any obstacle with two-dimensional or axial symmetry is related to the conventional viscous flow (at $m = \infty$) about the same obstacle by a simple transformation in which the velocity vector is multiplied by a negative constant. Thus the boundary layer increases in thickness from the rear stagnation point, separates, and culminates in a wake extending upstream from the nose (Fig. 1).

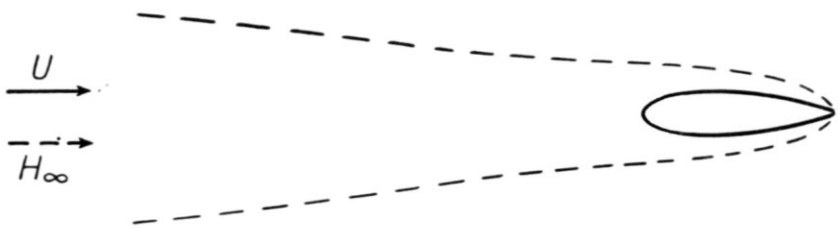

FIG. 1. Upstream wake and reversed boundary-layer growth.

At the opposite limit, moreover, i.e. when viscosity vanishes but electrical conductivity is finite, the downstream wake again disappears[4, 8, 11, 13, 14, 24], and only the upstream wake remains. This may be the more interesting of the two limiting cases, for real conducting fluids do, in fact, have relatively small viscosity and large electrical resistance. The significant dimensionless parameter in this comparison is the ratio of magnetic Reynolds number R_m to true Reynolds number Re, which is the "magnetic Prandtl number",

$$Pr_m = 4\pi\sigma\nu \tag{3}$$

As the name would imply, Pr_m is a material property. For liquid metals it has values of about 10^{-7}, while for ionized gases it varies with density and temperature and can be estimated by means of kinetic theory. Putting ν approximately equal to $a\lambda$, we have $Pr_m \approx 4\pi\sigma a\lambda$. The conductivity of air at about 5400°K and 0.015 times standard density is about 1 mho/cm[23]; hence Pr_m at these conditions is about 10^{-6}.

Thus, an appropriate limiting case as a model a real liquid or gaseous conductor is the case $Pr_m \to 0$. In this limit the two wakes can be identified unambiguously as a viscous wake whose thickness is $O(Re^{-1/2})$ and an inviscid wake whose thickness is $O(R_m^{-1/2})$, i.e. much thicker[8, 13, 14]. The former always extends rearward, while the latter extends upstream at sub-Alfvénic speeds.

Many of these remarks apply also to the boundary layer. For fluids of small Pr_m we have identified[5, 10] the phenomenon of an inviscid boundary layer, at least for bodies of sufficient thickness, and have noted that it is underlain by a viscous sublayer, $\sqrt{Pr_m}$ times as thick. This model seems to be in good agreement with what has been said above about wakes, and there are indications that this inviscid layer does increase in thickness from rear to front at sub-Alfvénic speeds, as would seem to be required.

2. *Upstream-inclined Waves*

It has already been mentioned above that wave-like disturbances occur in flows of conducting fluids, i.e. the Alfvén waves. In the general case of stream flow past an obstacle, such waves are produced, their orientation given simply by their propagation *relative to the fluid* at the Alfvén speed, along the magnetic lines. It is clear that various combinations of field direction and Alfvén number can result in some of these waves being inclined upstream.

The special case of aligned fields, however, is a degenerate one in this regard, for which recognizable Alfvén waves are not produced in an incompressible flow but are replaced by the wake-like phenomena described above. In a compressible fluid, on the other hand, Alfvén waves and conventional sound waves are replaces by families of combined acousti-magnetohydrodynamic waves in a rather complex way. Here again upstream-inclined waves—in fact, two families of them—may appear for certain field inclinations, Alfvén numbers, and Mach number[6,7,16], and now this category includes aligned-field flows at certain sub-Alfvénic speeds.

We therefore list this phenomenon, the occurrence of upstream-inclined acousti-mhd waves in steady aligned-field compressible flow, as another contrast between sub- and super-Alfvénic flow. In real fluids of small viscosity and finite conductivity, such waves are damped, i.e. diffused, but their orientations are substantially the same.

3. *Elliptic supersonic and hyperbolic subsonic flows*; *negative lift*

Finally, continuing our description of aligned-field compressible flow, we find a regime of supersonic flow, at Alfvén numbers less than 1, where wave phenomena disappear completely and the flow field is described by elliptic equations, as in conventional subsonic aerodynamics. On the other hand, again at sub-Alfvénic speeds, there is a regime of subsonic flow where the equations become hyperbolic and their solutions represent families of waves. Finally, in certain sub-Alfvénic regimes the surface pressures on obstacles are reversed in sign and the lift of airfoils at positive incidence might be expected to be negative[6,7,9,].

Having listed and briefly described these three different (but not necessarily unrelated) categories of sub-Alfvénic phenomena, we now proceed to a more detailed study of their origins to show that they do not violate physical principles.

ANALYSIS

In this section we shall attempt to explain the above-mentioned sub-Alfvénic phenomena, considering them in the same order and under the same headings.

1. *Upstream Wakes and Reversed Boundary Layers*

The momentum equation for an incompressible fluid is[1,13]

$$\mathbf{q} \cdot \nabla \mathbf{q} + \frac{1}{\varrho} \nabla p = \nu \nabla^2 \mathbf{q} + \frac{1}{4\pi\varrho} \left\{ \mathbf{H} \cdot \nabla \mathbf{H} - \frac{1}{2} \nabla H^2 \right\} \tag{4}$$

where the second right-hand term is the electromagnetic body force per unit mass, $\mathbf{j} \times \mathbf{H}/\varrho$, expressed with the aid of Ampére's Law, $4\pi\mathbf{j} = curl\mathbf{H}$. The second essential equation is Ohm's Law[1,3],

$$\mathbf{j} = \sigma\{\mathbf{E} + \mathbf{q} \times \mathbf{H}\} \tag{5}$$

or, after taking the curl of both sides,

$$\frac{1}{4\pi\sigma} \nabla^2 \mathbf{H} = \mathbf{q} \cdot \nabla \mathbf{H} - \mathbf{H} \cdot \nabla \mathbf{q} \tag{6}$$

It is usually convenient to combine the two gradient terms in Eq. (4) by defining the "total pressure" $P \equiv p + H^2/8\pi$; Eq. (4) can then be written in the from

$$\nu \nabla^2 \mathbf{q} = \mathbf{q} \cdot \nabla \mathbf{q} - \frac{1}{4\pi\varrho} \mathbf{H} \cdot \nabla \mathbf{H} + \frac{1}{\varrho} \nabla P \tag{7}$$

At this point it is clear, in Eqs. (6) and (7) that viscous diffusion of momentum and diffusion of magnetic field are involved; i.e. that ν and $1/4\pi\sigma$ are diffusion coefficients.

If we concentrate our attention on wake phenomena it is permissible to consider large distances from obstacles and to make appropriate approximations, particularly to small perturbations of the uniform velocity and magnetic fields. Furthermore, the usual boundary-layer or wake approximations, for a fluid of small viscosity lead to the conclusion that $\partial P/\partial y \approx 0$ and hence that in studies of wakes at large distances from obstacles the term $\varrho^{-1}(\partial P/\partial x)$ can be neglected in the x component of Eq. (7).

The results of these approximations are the following simplified equations:

$$U \frac{\partial u'}{\partial x} = \nu \nabla^2 u' + \frac{1}{4\pi\varrho} H_\infty \frac{\partial h_x}{\partial x} \tag{8}$$

$$U \frac{\partial h_x}{\partial x} = \frac{1}{4\pi\sigma} \nabla^2 h_x + H_\infty \frac{\partial u'}{\partial x} \tag{9}$$

Equation (8) states that the acceleration of a fluid particle is given by the usual diffusion of momentum by viscosity plus a term arising from the body force. Equation (9) states that the rate of increase of field strength in a fluid particle is due to the usual diffusion of the field plus the field intensification due to stretching of the fluid particle.

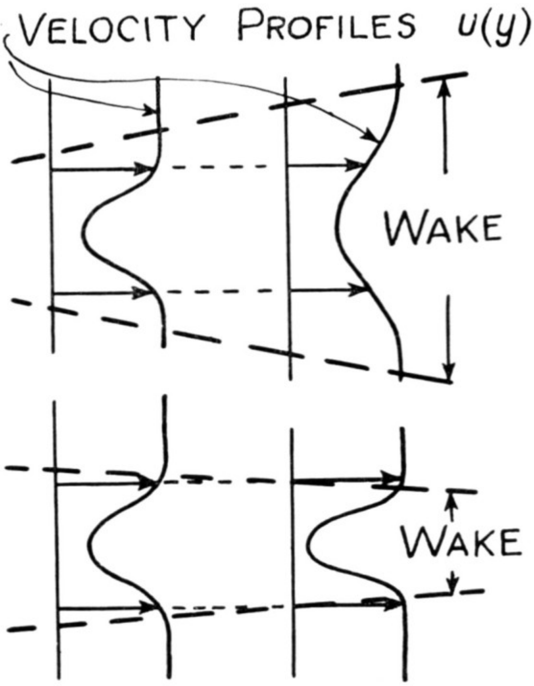

FIG. 2. Diagrams showing effects of viscous diffusion in conventional wake (above) and in sub-Alfvénic wake (below).

Now, to understand how a wake can extend upstream it is enough to consider the two limiting cases $Pr_m = \infty$ and $Pr_m = 0$; i.e. $\sigma = \infty$ and $\nu = 0$. Moreover, it is particularly interesting to study these cases, since for them the downstream wake disappears!

(a) $Pr_m = \infty\ (\sigma = \infty)$:

From Eq. (9),

$$U \frac{\partial h_x}{\partial x} = H_\infty \frac{\partial u'}{\partial x} \tag{10}$$

In fact, this is the case $\mathbf{q} \propto \mathbf{H}$; the flow and field vectors are parallel and proportional. But this makes the left-hand and second right-hand terms of Eq. (8) proportional to one another; the body force is proportional to the acceleration. The constant of proportionality is just m^{-2}, so that at sub-Alfvénic speed the body force dominates and the particle behaves as if it had negative mass.

Under these conditions it seems clear that the wake must extend upstream, or at least that it must diminish in width in the flow direction. A wake or boundary layer always involves outward diffusion of momentum-deficient particles (drag) near its edges, and this must accelerate rather than decelerate the fluid; thus the wake, if it extends upstream, diminishes in width toward the body (Fig. 2). Since the wake originates at the body, downstream it must collapse into a jet of zero thickness and disappear.

(b) $Pr_m = 0(\nu = 0)$

From Eq. (8):

$$U\frac{\partial u'}{\partial x} = \frac{H_\infty}{4\pi\varrho}\frac{\partial h_x}{\partial x} \tag{11}$$

But this makes the left-hand and second right-hand terms of Eq. (9) proportional; the field intensification due to stretching of the fluid is proportional to the total rate of increase of field strength $U\partial h_x/\partial x$. The constant of proportionality is again m^{-2}, so that at sub-Alfvénic speeds the "stretching" term dominates. Then magnetic-field diffusion results in an increase rather than a decrease of field strength in the U direction.

This reversed effect of diffusion is quite analogous to what we found in (a) for momentum diffusion; the wake in steady flow must therefore lie upstream of the body.

So far, we have discussed steady flows. It seems clear how the wake builds up in front of the body when the motion begins; namely by upstream Alfvén-wave propagation. To be sure, there is also downstream Alfvén-wave propagation, but the arguments presented above show that no downstream wake can result in steady flow, and they also describe the mechanism by which the transient disturbances downstream are eliminated as the steady state is approached.

Before leaving this subject let us consider briefly the effect of an upstream wake on the momentum balance of the problem. Fixing oneself in a coordinate system fixed to the body (Fig. 3), it would be easy to conclude that, since more momentum leaves the control zone through BB′ than enters through AA′, the drag must be negative. This, of course, ignores the body force; but the conclusion seems even more attractive when one determines the sign of the body forces, for they are directed upstream (see Fig. 3). The resolution of this dilemma comes from the fact that the pressure along AA′ is not uniform. Since the total pressure

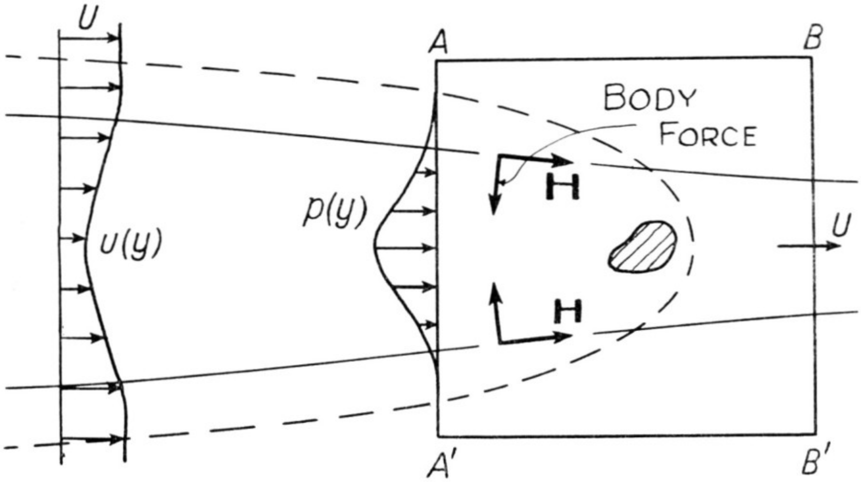

FIG. 3. Diagram showing control zone used for momentum balance in flow involving upstream wake.

P is uniform, the velocity deficiency along AA′ is accompanied by elevated pressures, which provide the necessary force in the downstream direction.

2. *Upstream-inclined Waves*

The wave patterns of compressible magnetohydrodynamic flow are best understood by use of what is called the "Friedrichs Diagram"(17) (Fig. 4). This is a diagram showing the self-perpetuating shape of a disturbance (pulse) produced at a point in a compressible ideal conductor otherwise at rest. It has a special orientation with respect to the magnetic-field vector **H**. It is analogous to the circular disturbance pattern of ordinary acoustics but differs from it in an essential way, for it consists not only of a wave front (fast wave) but of certain slower discontinuities (crests) which lie within the disturbed region (slow waves). (There may also be intermediate waves, which are not involved in the present discussion.)

Just as the Mach waves of ordinary supersonic flow can be explained as the envelope of circular disturbances produced by a moving body, the standing acousti-mhd waves are the envelopes of fast or slow waves of the Friedrichs pulses made by the moving body. These waves can be inclined upstream if the body's speed is such that an upstream envelope is appropriate. Such a case, with aligned fields at infinity, is sketched in Fig. 5; it belongs to a category of subsonic, sub-Alfvénic flows which will be specified below. Upstream-inclined waves are possible only if

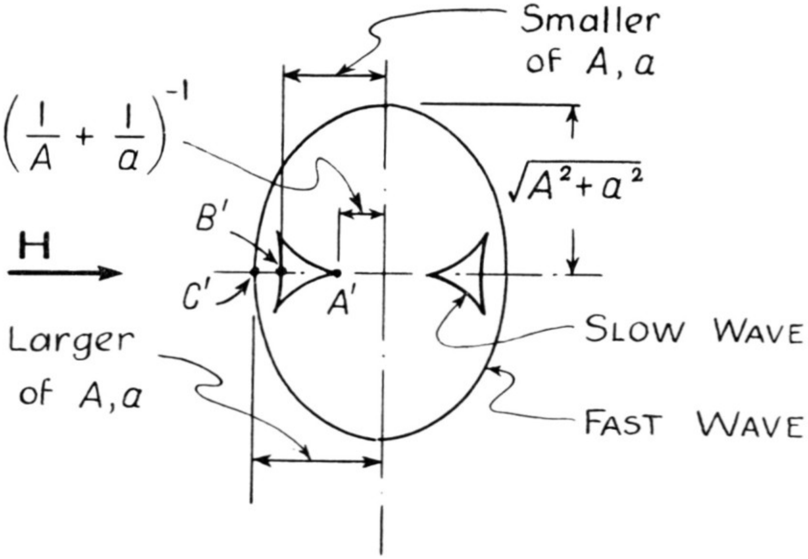

FIG. 4. "Friedrichs Diagram" of self-similar pulse shape due to point disturbances in perfectly conducting inviscid compressible fluid otherwise at rest. A is the Alfvén wave speed and a the sound speed.

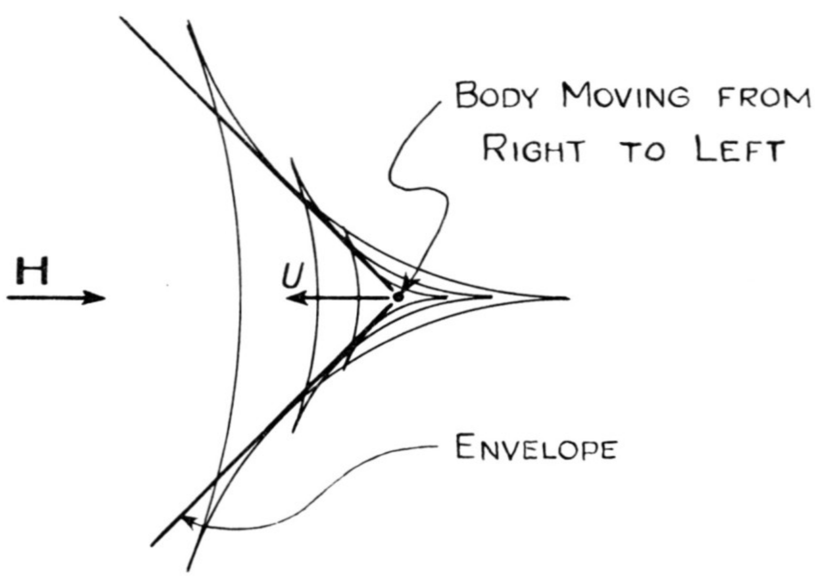

FIG. 5. Sketch showing how a body moving from right to left forms upstream-inclined envelopes of the "slow waves" of the Friedrichs diagram.

the speed of the body lies between the propagation speeds of points A′ and B′ on the pulse diagram (Fig. 4).

It might be noted that the differential equation for steady flow in this regime is, for plane flow[7, 10],

$$(\mathcal{M}^2-1)\psi_{xx} = \psi_{yy} \tag{12}$$

where

$$\mathcal{M}^2 = \frac{M^2m^2}{M^2+m^2-1} \tag{13}$$

and $\mathcal{M}^2 < 1$ in the particular regime represented in Fig. 5. Thus the general solution of Eq. (12) is

$$\psi(x, y) = f(x-By)+g(x+By) \tag{14}$$

where $B^2 = \mathcal{M}^2-1$ and the arbitrary functions f and g represent waves of inclination $\pm 1/B$.

The choice between the waves in a given circumstance, as always, must be made on physical grounds; i.e. one must choose outgoing, rather than incoming waves, if one is treating unconfined flow. This selection can be facilitated, however, by a general principle that makes use of the properties of the steady-flow equations and does not require such detailed knowledge about the time-dependent solution as has been available above. The principle can be stated as follows: In any regime where the slope of characteristics ("Mach waves") increases with increasing flow speed, the physically correct family of waves is upstream-inclined, and vice versa. A study of Eq. (13) will reveal that this does indeed describe the regime $m < 1$, $1-m^2 < M^2 < 1$, in the present problem.

This principle follows from what has already been said about standing waves (characteristics) as envelopes of disturbances propagating from successive points of the flight path. Increased flight speed can only steepen the waves if the envelopes are formed ahead of the body, as in Fig. 5. Conversely, if increased flight speed reduces the wave inclination, the envelopes must lie behind the body as in conventional supersonic aerodynamics.

In this light it becomes clear that upstream-inclined waves in steady-flow problems are not just a consequence of an upstream-propagation mechanism, such as Alfvén waves, for such a mechanism always exists at subcritical stream speeds. Rather, they are the result of a peculiar pulse-propagation shape that is capable of forming envelopes ahead of a body. It seems possible, therefore, that upstream-inclined waves may occur in the physical situations involving modification of conventional aerodynamics due to anisotropic propagation mechanisms.

Before leaving the subject of upstream-inclined waves, let us consider two simple examples of flow and pressure patterns in this regime. In Fig. 6 is shown plane flow about a double-wedge profile for infinite conductivity and vanishing viscosity. The sketch is drawn according to small-perturbation theory; if nonlinear terms were retained the foremost

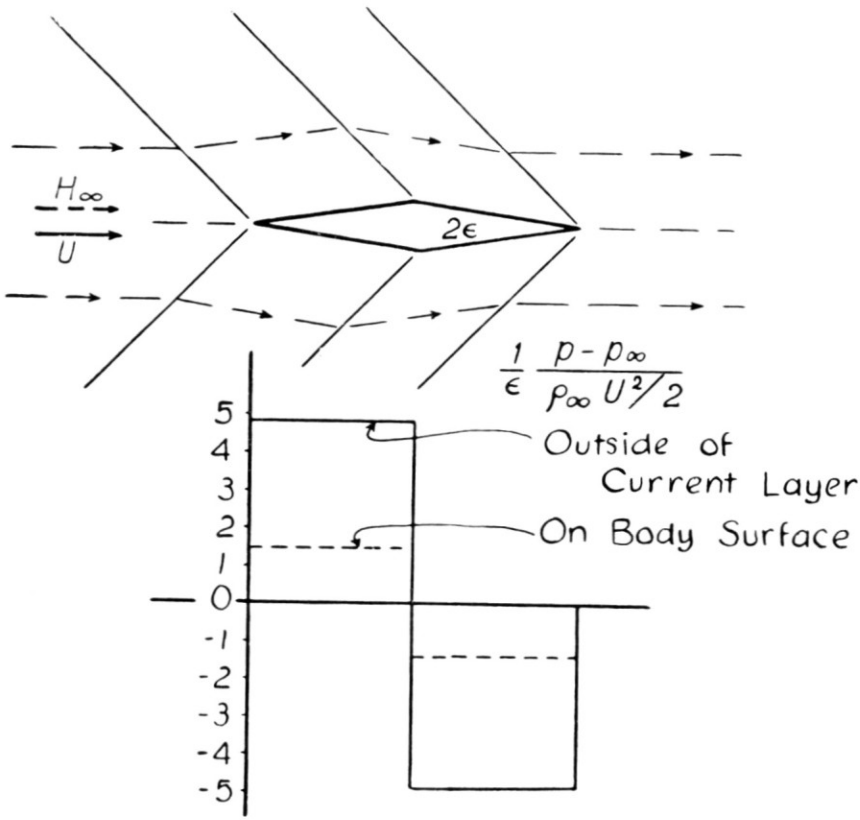

FIG. 6. Wave and streamline pattern for plane flow past a double-wedge profile at $M = m = 0{\cdot}77$, $\mathcal{M} = \sqrt{2}$

and rearmost waves would be shock waves and the central waves would be replaced by expansion fans. Magnetohydrodynamic shock waves have been treated by several authors[(18–21)] and will not be discussed here, except to emphasize that the upstream-inclined acousti-mhd waves discussed in this section do have their counterpart in upstream-inclined shocks, in this same regime of sub-Alfvénic, subsonic flow.

In further explanation of Fig. 6 it should also be emphasized that the current sheets at the body surface are an infinite-conductivity approximation to the high-current-density magnetic boundary layers that actually exist there in fluids of finite conductivity, and which were mentioned in our Introduction. The vanishing of magnetic field within the body is an inevitable consequence of infinite conductivity, assuming only that the body is not a perfect conductor. In a fluid of finite conductivity there remains a residual field within the body, of magnitude $O(H_\infty R_m^{-1/2})$. Thus the magnetic boundary layer, which replaces the current sheet, makes the transition from field strength $O(H_\infty)$ outside to $O(H_\infty R_m^{-1/2})$ inside. It will be seen (Fig. 6) that in this regime the change of pressure across this thin layer is large.

As a second example, Fig. 7 shows flow of a perfect conductor ($\gamma = {}^5/_3$) around a corner in this regime. The flow is quite analogous to Prandtl–Meyer flow, except that the waves are upstream-inclined. It is also remarkable that although $\mathcal{M}$ goes from ∞ to 1 in this Figure, the corresponding velocities, pressures, densities, and temperatures are everywhere finite. The case sketched represents expansion through the maximum angle for this isentrope; further turning in the same direction beyond B would be an elliptical process, in spite of its supersonic speed, as will be mentioned again below. Thus additional turning, either upstream of A or downstream of B would render the problem a mixed hyperbolic–elliptic one.

3. *Elliptic Supersonic and Hyperbolic Subsonic Flows*; *Negative Lift*

Turning now to the third category of sub-Alfvénic phenomena, we refer once again to the Friedrichs Diagram (Fig. 4). If a body moves parallel or anti-parallel to $\mathbf{H}$ at a speed between the propagation speeds of points B' and C', it can form no standing waves, and the corresponding steady-flow pattern is elliptical, even though this speed may be supersonic. Referring to Eqs. (12) and (13) above, we find that this is the range $m < 1$, $1 < M < A/a$ provided the ratio of disturbance speeds, A/a, is greater than 1. Here $\mathcal{M} < 1$.

This is surely a most remarkable regime of flow, where the streamline pattern is described by elliptical equations, which can be obtained from an incompressible irrotational flow pattern by a simple Prandtl–Glauert transformation as suggested by Eq. (12), but where the flow is supersonic! In the approximation of a perfect conductor the flow is isentropic. Thus the variation of flow speed with stream-tube area is the reverse of subsonic flow: the speed is high where the streamline spacing is wide, and vice versa. One result is that the lift of an airfoil at positive incidence is negative, and this conclusion is not altered by consideration of the surface-current forces, for they augment the surface pressures in this region. This conclusion, of course, is based on the presumption that the incompressible flow pattern including circulation can be carried over to the compressible flow in the Prandtl–Glauert transformation. Since the circulation is actually the result of some complex viscous phenomena at the trailing edge it may be a gross oversimplification to assume that it transforms this way. Thus the conclusion involving negative lift must be regarded as speculative until experimental evidence is available.

The regime of hyperbolic subsonic flow has already been discussed. It might be mentioned that its nature is just the opposite of the elliptic supersonic regime just described; namely, it involves streamline patterns (e.g. Figs. 6 and 7) resembling supersonic flow, but the velocity and

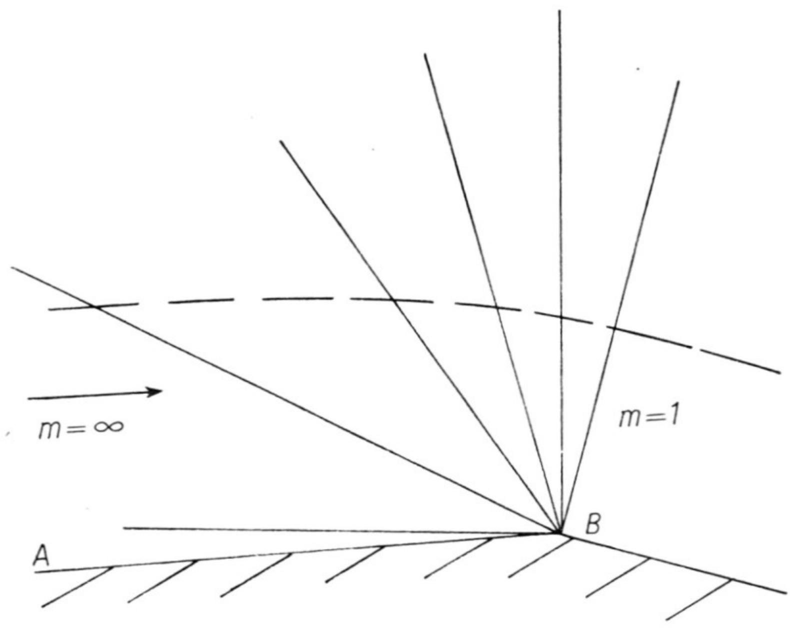

FIG. 7. Simple-wave flow about a corner in the hyperbolic, subsonic, sub-Alfvénic regime. For the case shown, the upstream conditions (A) are $M = 0.565$, $m = 0.83$, $\mathcal{M} = \infty$ and the downstream conditions (B) are $M = \mathcal{M} = 1$. The angle turned through is about 18°. $\gamma = 5/3$.

pressure variation within these stream tubes is typically subsonic. It is clear in Figs. 6 and 7 how this is accomplished: the appearance of up-stream-inclined waves yields streamlines whose spacing is appropriate for subsonic flow, i.e. wider where speeds are low and narrower where speeds are high. Although the forces on the surface-current layers in this regime oppose normal lift (cf. Fig. 6), they do not dominate the situation, and this is not a regime of reversed lift.

Finally, however, there is a third area of sub-Alfvénic flow, namely the range of flow speeds less than the speed of point A' in Fig. 4. This is a second region of elliptic flow patterns; viz. (See Eqs. (12) and (13))

$m < 1, m^2+M^2 < 1$, and $\mathcal{M}^2 < 0$. This too, is a remarkable flow regime, for it is related to incompressible irrotational flow by means of a Prandtl–Glauert transformation to a compressed, rather than stretched, plane. Here the surface-current forces dominate the fluid pressures, and net pressure on the body surface is reversed in sign as a result. Thus this is a second region of negative lift, although the reason is not quite the same as in the elliptic supersonic regime discussed above. But once again the conclusion of negative lift is based on the presumption that flow with circulation is related uniquely to an incompressible irrotational flow in the transformation. This presumption seems particularly dangerous in the present case since this is precisely one of the regimes of upstream-wake flow described above.

Recognizing that in a real conducting fluid there would be an upstream wake, largely inviscid, and a thinner downstream wake, largely viscous, is one safe in assuming that conventional viscous processes at the trailing edge control the circulation? Should the inviscid wake dominate—which seems possible in view of its greater thickness—the "Kutta–Joukowski condition" might well be shifted to the leading edge[(4)]. Circulation would then be reversed in sign, for given incidence, and the net result, with surface currents (or their real-fluid counterparts), would be positive lift at positive incidence. Again one must defer to experimental observations to resolve this question, unless the whole process of transient flow and separation can be calculated for a fluid of real viscosity and conductivity.

These speculations regarding the Kutta–Joukowski condition and circulation in sub-Alfvénic flow are made more intriguing by a recent investigation by Ring[(8)] of unsteady magnetohydrodynamic flow past airfoils. One of Ring's conclusions is that such flow is unstable, in the sense that divergent oscillations of circulation can occur, unless the conventional trailing-edge (Kutta–Joukowski) condition is discarded.

Finally, it might be pointed out that negative drag is never predicted by the theories discussed here, in spite of the reversal of net surface pressure in certain regimes, for these are always regimes of elliptic flow patterns where the pressure drag vanishes.

There remains, however, one more possibility to be discussed: what is the likelihood that all of the sub-Alfvénic phenomena predicted and discussed here are simply results of having postulated the entire flow pattern incorrectly? At least one author (Stewartson[(22)]) has suggested that entirely different flow patterns exist, and has provided evidence to support his view.

Stewartson believes that the flows described here, and in numerous other papers referred to here, are incorrect by virtue of the familiar assumption of undisturbed flow far from an obstacle. He points out that a different flow pattern may occur, in which the parallel magnetic field is undisturbed, the fluid is at rest between parallel vortex sheets tangent to the body at its top and bottom, and the fluid stream outside of these vortex sheets is parallel and undisturbed. Moreover, by treating transient flow in the limit of infinite conductivity and $M = m = 0$, he has shown that this is the ultimate flow pattern for large time.

Although the Stewartson flow undoubtedly satisfies the boundary conditions of the problem (except the rejected condition at infinity), we believe that it is a very special situation that pertains only to the limit $m \to 0$. At this limit the magnetic lines are infinitely rigid compared to any inertia or pressure forces that the fluid can exert. Thus, when the motion begins, the fluid is inevitably constrained to move only along the magnetic lines, and Stewartson flow results. It is difficult to see how a similar pattern could result if m were increased appreciably, so that fluid motion around the body could occur.

We note that there are other situations where a stabilizing body force results in analogous flow-between-rigid-vortex-sheets in the limit of very slow motion. These are flow due to motion of a body in (a) a rotating liquid and (b) a highly stratified liquid in a gravitational field. Yet one does not find these patterns approximated at higher speeds. Thus we expect that Stewartson flow will occur near $m = 0$, for sufficiently large R_m, but that the categories of flow discussed here will be observed in a larger range of values of the Alfvén number. The resolution of this problem stands as a most attractive problem for experimenters.

CONCLUSIONS

In the last section of this paper we propose to discuss briefly the prospects of laboratory observation of the various sub-Alfvénic phenomena described above.

There is, of course, no problem in making the Alfvén number m as small as desired, since this can be done either by reducing the flow speed or increasing the field strength. What is more critical is to obtain simultaneously sub-Alfvénic values of m and large enough R_m to bring out these phenomena, most of which we have predicted for fluids of large conductivity. To be more specific: at low values of R_m the upstream wake may be so diffuse as to be unrecognizable, and the same is true of upstream-inclined waves. Moreover, since conventional Mach waves surely appear in supersonic flow at small enough values of R_m, the disappearance of waves in elliptic supersonic flow must be to some degree a high-conductivity phenomenon. The question arises: what laboratory values of R_m are required to affect substantially the waves in this flow regime?

To help answer these questions we have carried out the following estimates:

Upstream Wakes

Using the results of Lary[4] (equation 186) one can easily estimate the perturbation velocity on the axis ahead of a slender body of revolution with a forward-facing wake. In this case the velocity that would occur without any magnetohydrodynamic interaction must be multiplied by the factor $\left(1+\frac{R_m}{4\pi m^2}\right)$; thus if $m = \frac{1}{2}$ an R_m of 4π would be required to obtain a disturbance five times as great as the disturbance that would occur with no magnetohydrodynamic interaction at a distance of two body lengths upstream of the nose. Such a difference in flow pattern should be easily discernible.

Subsonic Hyperbolic Flow

In this regime of flow the magnetic Reynolds number necessary for forward-facing waves generated at a wavy wall to propagate a wave length into the fluid away from the wall before the disturbance is damped by a factor $1/e$ is given by the expression[25]

$$R_m = \frac{-2\pi \cos \varphi}{\sin^2\varphi\,(1-m^2)} \tag{15}$$

where φ is the angle the wave makes with the free stream ($\varphi < 90°$ in this regime). Note that R_m is least when $\varphi = 90°$ but then the wave coincides with an ordinary sonic wave front. For a wave swept forward 15°, or $\varphi = 105°$, and $m = \frac{1}{2}$, an R_m equal to 2.59 is required. In this case the flow is only slightly subsonic, the Mach number being 0.987.

Supersonic Elliptic Flow

In this type flow the ordinary acoustic waves disappear as R_m increases if $m < 1$. The disturbances carried along waves generated again by a wavy wall, that ordinarily in the absence of magnetohydrodynamic effects would extend to infinity, are damped to $1/e$ of their wall value a wave-length away from the wall at an R_m given by[25]:

$$R_m = 2m^2 \tan \varphi \tag{16}$$

Therefore if again $m = \frac{1}{2}$ and for $\varphi = 45°$, $R_m = \frac{1}{2}$.

All of the R_m computed in the three instances can be achieved experimentally in the laboratory. All of the sub-Alfvénic phenomena discussed should be capable of experimental verification.

LIST OF SYMBOLS

A	speed of Alfvén waves, $\mathbf{H}/\sqrt{4\pi\varrho_{1/2}}$
a	speed of sound waves, $(dp/d\varrho)^{\frac{1}{2}}$ isentropic
B	$(\mathcal{M}^2-1)^{1/2}$
$\mathbf{E}$	electric-field vector
f, g	arbitrary functions (Eq. (14))
$\mathbf{H}$	magnetic-field vector
H	scalar magnitude of $\mathbf{H}$
H_∞	scalar magnitude of $\mathbf{H}$ at infinity
h_x, h_y	perturbation components of the magnetic-field vector
$\mathbf{j}$	electric-current density
L	reference length for Reynolds numbers
M	free-stream Mach number, U/a
$\mathcal{M}$	(See Eq. (13))
m	Alfvén number, U/A
P	total pressure, $p+H^2/8\pi$
Pr_m	magnetic Prandtl number, $4\pi\sigma\nu$
p	static pressure
$\mathbf{q}$	fluid-velocity vector
Re	Reynolds number, UL/ν
R_m	magnetic Reynolds number, $4\pi UL\sigma$
U	free-stream flow speed
u', v'	perturbation components of the fluid-velocity vector
x, y	plane Cartesian coordinates
γ	ratio of specific heats
λ	mean free path of molecules
ν	kinematic viscosity
ϱ	mass density of fluid
σ	electrical conductivity
ψ	perturbation stream function; total stream function $= Uy+\psi$
ε	semi-wedge angle in Fig. 6
φ	wave angle

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